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Translated by J. J. D.

UDC 531.36

STABILITY OF UNSTEADY MOTIONS IN FIRST APPROXIMATION

PMM Vol. 40, № 3, 1976, 424-430
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(Received April 18, 1975)

Results obtained in [1] are extended to nonautonomous systems and a wider class of nonlinearities. The question of application of the Liapunov vector function is considered.

1. Let us consider the system of differential equations of perturbed motion

$$\mathbf{y}^{\cdot} = Q(t)\mathbf{x} + R(t)\mathbf{y} + \mathbf{Y}^{\circ}(t, \mathbf{y}) + \mathbf{Y}(t, \mathbf{x}, \mathbf{y}) \tag{1.1}$$
$$\mathbf{x}^{\cdot} = P(t)\mathbf{x} + \mathbf{X}(t, \mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \mathbf{R}^n, \quad \mathbf{y} \in \mathbf{R}^k$$

where P, Q and R are continuous and bounded for $t \geqslant 0$ matrices of corresponding order and functions $\mathbf{Y}$ and $\mathbf{X}$ satisfy conditions

$$\mathbf{Y}(t, 0, \mathbf{y}) \equiv 0, \quad \mathbf{X}(t, 0, \mathbf{y}) \equiv 0 \tag{1.2}$$

$$\frac{\|\mathbf{Y}(t, \mathbf{x}, \mathbf{y})\| + \|\mathbf{X}(t, \mathbf{x}, \mathbf{y})\|}{\|\mathbf{x}\|} \underset{t \geqslant 0}{\rightrightarrows} 0 \quad \text{for } \|\mathbf{x}\| + \|\mathbf{y}\| \to 0 \tag{1.3}$$

We assume that solutions of the linear system

$$\mathbf{x}^{*\cdot} = P(t)\mathbf{x}^* \tag{1.4}$$

satisfy the condition

$$\|\mathbf{x}^*(t; t_0, \mathbf{x}_0^*)\| \leqslant B\|\mathbf{x}_0^*\| e^{-\alpha(t-t_0)} \quad (B > 0, \ \alpha > 0\text{-const}; \ t \geqslant t_0 \geqslant 0) \tag{1.5}$$

Let us consider the system

$$\mathbf{y}^{*\cdot} = R(t)\mathbf{y}^* + \mathbf{Y}^{\circ}(t, \mathbf{y}^*) \tag{1.6}$$

which is obtained from the first group of Eqs. (1.1) for $\mathbf{x} = 0$ whose solutions are denoted by $\mathbf{y}^*(t; t_0, \mathbf{y}_0^*)$. The variational equations for system (1.6) are of the form

$$\xi^{\cdot} = \left[R(t) + \frac{\partial \mathbf{Y}^{\circ}(t, \mathbf{y}^*)}{\partial \mathbf{y}^*}\Big|_{\mathbf{y}^*=\mathbf{y}^*(t;\, t_0,\, \mathbf{y}_0^*)}\right] \xi \tag{1.7}$$

We denote by $\Omega(t; t_0, \mathbf{y}_0^*)$ the fundamental matrix of solutions of system (1.7); we have $\Omega(t_0; t_0, \mathbf{y}_0^*) = E$, where E is the unit matrix, and $\Omega(t; t_0, \mathbf{y}_0^*) = \partial \mathbf{y}^*(t; t_0, \mathbf{y}_0^*) / \partial \mathbf{y}_0^*$.

2. Theorem 1. Let us assume that

1) the zero solution of system (1.4) is exponentially asymptotically stable (see (1.5));

2) the zero solution of system (1.6) is uniformly stable relative to t_0, and

3) there exist constants $h > 0$ and $N > 0$ such that for $t_0 \geqslant 0$ and $\|\mathbf{y}_0^*\| \leqslant h$ we have

$$\|\Omega(t; t_0, \mathbf{y}_0^*)\| \leqslant N \quad \text{for} \quad t \geqslant t_0 \tag{2.1}$$

The unperturbed motion $\|\mathbf{x}\| = \|\mathbf{y}\| = 0$ of system (1.1) is then Liapunov-stable, and exponentially asymptotically $\mathbf{x}$-stable for any functions $\mathbf{Y}$ and $\mathbf{X}$ that satisfy conditions (1.2) and (1.3).

Proof. As shown in [2], condition (1.5) implies the existence of a positive definite quadratic form $V(t, \mathbf{x})$ with bounded coefficients, which satisfies the equation

$$\partial V(t, \mathbf{x}) / \partial t + \operatorname{grad}_{\mathbf{x}} V(t, \mathbf{x}) \cdot P(t)\mathbf{x} = -\|\mathbf{x}\|^2$$

The second group of Eqs. (1.1) implies that the derivative of function V is of the form

$$V^{\cdot}(t, \mathbf{x}, \mathbf{y}) = -\|\mathbf{x}\|^2 + \operatorname{grad}_{\mathbf{x}} V(t, \mathbf{x}) \cdot \mathbf{X}(t, \mathbf{x}, \mathbf{y}) \tag{2.2}$$

According to (1.3) there exists such β, $0 < \beta < h$, that in the region

$$t \geqslant 0, \quad \|\mathbf{x}\| \leqslant \beta, \quad \|\mathbf{y}\| \leqslant \beta \tag{2.3}$$

the inequality

$$V^{\cdot}(t, \mathbf{x}, \mathbf{y}) \leqslant -c_1\|\mathbf{x}\|^2 \quad (c_1 = \text{const} > 0) \tag{2.4}$$

is satisfied.

Let us consider the arbitrary solution $\mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{y}_0)$, $\mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0)$ of system (1.1) with initial conditions in the region

$$t_0 \geqslant 0, \quad \|\mathbf{x}_0\| < \delta, \quad \|\mathbf{y}_0\| < \delta, \quad \delta < \beta \tag{2.5}$$

That solution satisfies conditions

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{y}_0)\| \leqslant \beta, \quad \|\mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0)\| \leqslant \beta \tag{2.6}$$

at least in some interval (t_0, T). Hence, by virtue of (2.4) we have [3]

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{y}_0)\| \leqslant c_2\|x_0\| e^{-\gamma(t-t_0)} \quad \text{for} \quad t \in (t_0, T) \quad (c_2 > 0,\ \gamma > 0\text{-const}) \tag{2.7}$$

Condition (1.3) and the inequality (2.7) yield the estimate

$$\|\mathbf{Y}(t, \mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0))\| \leqslant c_3\|\mathbf{x}_0\| e^{-\gamma(t-t_0)} \tag{2.8}$$
$$t \in (t_0, T), \quad c_3 = \text{const} > 0$$

Function $\mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0)$ which satisfies the system of equations

$$\mathbf{y}^{\cdot} = R(t)\mathbf{y} + \mathbf{Y}^{\circ}(t, \mathbf{y}) + Q(t)\mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{y}_0) + \mathbf{Y}(t, \mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0))$$

can be represented by the formula [4]

$$\mathrm{y}(t;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0) = \mathrm{y}^*(t;\, t_0,\, \mathrm{y}_0) + \int_{t_0}^{t} \Omega(t;\, \tau,\, \mathrm{y}(\tau;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0)) \times \tag{2.9}$$

$$[Q(\tau)\mathrm{x}(\tau;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0) + \mathrm{Y}(\tau,\, \mathrm{x}(\tau;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0),\, \mathrm{y}(\tau;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0))]\, d\tau$$

By stipulation

$$\|Q(t)\| \leqslant M = \mathrm{const} \quad \text{for } t \geqslant 0 \tag{2.10}$$

Using (2.7), (2.8), (2.10) and condition (3) of the theorem, for $t \in (t_0,\, T)$ from (2.9) we obtain

$$\|\mathrm{y}(t;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0)\| \leqslant \|\mathrm{y}^*(t;\, t_0,\, \mathrm{y}_0)\| + \tag{2.11}$$

$$N(Mc_2 + c_3)\|\mathrm{x}_0\| \int_{t_0}^{t} e^{-\gamma(\tau - t_0)} d\tau \leqslant \|\mathrm{y}^*(t;\, t_0,\, \mathrm{y}_0)\| + N(Mc_2 + c_3)\gamma^{-1}\|\mathrm{x}_0\|$$

Let ε be an arbitrarily small number, $0 < \varepsilon < \beta$. We set $\delta(\varepsilon) > 0$ in (2.5) sufficiently small to have the inequalities

$$\delta < {}^1/_2 c_4 \varepsilon, \quad c_4 = \min\{c_2^{-1},\, \gamma N^{-1}(Mc_2 + c_3)^{-1}\} \tag{2.12}$$

$$\|\mathrm{y}^*(t;\, t_0,\, \mathrm{y}_0)\| < {}^1/_2 \varepsilon \quad \text{for} \quad t \geqslant t_0,\, t_0 \geqslant 0 \tag{2.13}$$

satisfied (the latter is possible by virtue of condition (2) of the theorem). Then from (2.7) and (2.11) it follows that for $t \in (t_0,\, T)$:

$$\|\mathrm{x}(t;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0)\| < \varepsilon e^{-\gamma(t - t_0)}, \quad \|\mathrm{y}(t;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0)\| < \varepsilon \tag{2.14}$$

Thus the inequalities (2.14) are valid throughout the time interval during which conditions (2.6) are satisfied. Since $\varepsilon < \beta$, the inequalities (2.14) are valid for all $t \geqslant t_0$. The theorem is proved.

Note. The above proof shows that condition (3) of Theorem 1 ensures the uniform stability of the zero solution of system (1.6) with respect to t_0 in the case of constantly acting perturbations $\mathrm{r}(t,\, \mathrm{y})$ which in region $t \geqslant t_0$, $\|\mathrm{y}\| \leqslant \beta > 0$ satisfy the inequality $\|\mathrm{r}(t,\, \mathrm{y})\| \leqslant \delta \exp[-\gamma(t - t_0)]$, where $\delta > 0$ is arbitrarily small and $\gamma = \mathrm{const} > 0$.

3. Condition (3) of Theorem 1 is difficult to prove. It is possible to dispense with that condition.

Theorem 2. Let us assume that:

1) the zero solution of system (1.4) is exponentially asymptotically stable;

2) the zero solution of the system

$$\mathrm{y}^{*\cdot} = R(t)\mathrm{y}^* \tag{3.1}$$

is uniformly stable with respect to t_0, and

3) function $\mathrm{Y}^\circ(t,\, \mathrm{y})$ satisfies conditions

$$\|\mathrm{Y}^\circ(t,\, \mathrm{y})\| \leqslant \varphi(t)\|\mathrm{y}\|, \quad \int_0^\infty \varphi(t)\, dt = D < \infty \tag{3.2}$$

The unperturbed motion of system (1.1) is then by Liapunov uniformly stable and exponentially asymptotically x-stable for any functions Y and X that satisfy conditions (1.2) and (1.3).

Proof. The initial part of the proof of Theorem 1 up to and including the estimate (2.8) remains in this case unchanged. Function $\mathrm{y}(t;\, t_0,\, \mathrm{x}_0,\, \mathrm{y}_0)$, considered as the solution of system

$$\mathbf{y}^{\boldsymbol{\cdot}} = R(t)\,\mathbf{y} + \mathbf{Y}^{\circ}(t,\ \mathbf{y}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0)) + Q(t)\,\mathbf{x}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0) + \mathbf{Y}(t,\ \mathbf{x}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0),\ \ \mathbf{y}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0))$$

can be represented by the Cauchy formula [5]

$$\mathbf{y}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0) = \Omega(t;\ t_0)\,\mathbf{y}_0 + \int_{t_0}^{t} \Omega(t;\ \tau)\,[\mathbf{Y}^{\circ}(\tau;\ \mathbf{y}(\tau;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0)) + Q(\tau)\,\mathbf{x}(\tau;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0) + \mathbf{Y}(\tau,\ \mathbf{x}(\tau;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0),\ \ \mathbf{y}(\tau;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0))]\,d\tau \qquad (3.3)$$

where $\Omega\ (t;\ \ t_0)$ is the fundamental matrix of the solution of system (3. 1), and $\Omega\ (t_0;\ t_0) = E$. Condition (2) of the theorem is equivalent to the inequality [5]

$$\|\Omega(t;\ t_0)\| \leqslant N = \text{const} \quad \text{for} \quad t \geqslant t_0,\ t_0 \geqslant 0 \qquad (3.4)$$

Using (3. 4), (3. 2), (2. 7), (2. 8) and (2. 10), from (3. 3) we have

$$\|\mathbf{y}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0)\| \leqslant N\|\mathbf{y}_0\| + N\int_{t_0}^{t} \varphi(\tau)\,\|\mathbf{y}(\tau;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0)\|\,d\tau + N(Mc_2 + c_3)\,\|\mathbf{x}_0\|\int_{t_0}^{t} e^{-\gamma(\tau-t_0)}d\tau \leqslant N\,[\|\mathbf{y}_0\| + (Mc_2 + c_3)\,\gamma^{-1}\|\mathbf{x}_0\|\,] + N\int_{t_0}^{t} \varphi(\tau)\,\|\mathbf{y}(\tau;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0)\|\,d\tau, \quad t \in (t_0,\ T)$$

Applying to the last inequality the Gronwall-Bellman lemma [5, 6] and taking into account (3. 2), we obtain

$$\|\mathbf{y}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0)\| \leqslant N\,[\|\mathbf{y}_0\| + (Mc_2 + c_3)\,\gamma^{-1}\|\mathbf{x}_0\|\,] \times \exp\left[N\int_{t_0}^{t} \varphi(\tau)\,d\tau\right] \leqslant N\,[\|\mathbf{y}_0\| + (Mc_2 + c_3)\,\gamma^{-1}\|\mathbf{x}_0\|\,]\,e^{DN}$$

The proof is completed similarly to that of Theorem 1.

Note. The uniform stability of the zero solution of system (1. 6) follows from conditions (2) and (3) of Theorem 2. It is proved similarly to the foregoing with the use of the Cauchy formula and the Gronwall-Bellman lemma.

Let $\mathbf{Y}^{\circ}\ (t,\ \mathbf{y}) \equiv 0$; for which systems (1. 6) and (1, 1) assume, respectively, the form (3. 1) and

$$\mathbf{y}^{\boldsymbol{\cdot}} = Q(t)\,\mathbf{x} + R(t)\,\mathbf{y} + \mathbf{Y}(t,\ \mathbf{x},\ \mathbf{y}), \quad \mathbf{x}^{\boldsymbol{\cdot}} = P(t)\,\mathbf{x} + \mathbf{X}(t,\ \mathbf{x},\ \mathbf{y}) \quad (3.5)$$

The following theorem results from Theorem 2 for $\varphi\ (t) \equiv 0$.

Theorem 3. Let us assume that the zero solution of system (1. 4) is exponentially asymptotically stable and the zero solution of system (3. 1) is uniformly stable with respect to t_0. Then the unperturbed motion of system (3. 5) is uniformly Liapunov-stable and exponentially asymptotically $\mathbf{x}$-stable for any functions $\mathbf{Y}$ and $\mathbf{X}$ that satisfy conditions (1. 2) and (1. 3).

Note. (1) Theorem 3 ceases to be valid when the stability of the zero solution of system (3. 1) is nonuniform with respect to t_0, as shown by the example constructed by Perron [2, 6].

2) When matrices P, Q and R are constant, Theorem 3 is the same as Theorem 1

in [1].

It is possible to disregard condition (3) of Theorem 1, when the uniform stability of the zero solution of system (1.6) (see condition (2) of Theorem 1) is established by means of the Liapunov function with suitable properties.

Let ω and ψ be vectors in the space $\mathbf{R}^m$. For $\omega_i \leqslant \psi_i$ $(i = 1, \ldots, m)$ we have $\omega \leqslant \psi$.

Let us assume that there exists a vector function $\mathbf{v}(t, \mathbf{y}^*) = (v_1(t, \mathbf{y}^*), \ldots, v_m(t, \mathbf{y}^*))$ such that:

1) $\mathbf{v}$ and the derivative $\mathbf{v}^\cdot$ are continuous by virtue of system (1.6) and $\mathbf{v}(t, 0) \equiv \mathbf{v}^\cdot(t, 0) \equiv 0$;

2) for some l, $1 \leqslant l \leqslant m$, $v_1 \geqslant 0, \ldots, v_l \geqslant 0$ and

$$v_1(t, \mathbf{y}^*) + \ldots + v_l(t, \mathbf{y}^*) \geqslant a(\|\mathbf{y}^*\|)$$

where $a(r)$ is a continious monotonically increasing function and $a(0) = 0$;

3) the partial derivatives $\partial \mathbf{v}/\partial \mathbf{y}^*$ satisfy conditions

$$\|\partial \mathbf{v} / \partial \mathbf{y}^*\| \leqslant \varphi(t) \tag{3.6}$$

$$\int_{t_0}^{\infty} \varphi(t) e^{-\gamma(t-t_0)} dt < D = \text{const} \qquad \text{for all} \quad t_0 \geqslant 0 \tag{3.7}$$

4) the derivative $\mathbf{v}^\cdot$ satisfies by virtue of (1.6) the inequality

$$\mathbf{v}^\cdot(t, \mathbf{y}^*) \leqslant \mathbf{f}(t, \mathbf{v}(t, \mathbf{y}^*)) \tag{3.8}$$

5) the vector function $\mathbf{f}(t, \mathbf{v})$ is determinate and continuous in the region

$$t \geqslant 0, \ \|\mathbf{v}\| < R, \quad v_1 \geqslant 0, \ldots v_l \geqslant 0 \tag{3.9}$$

where $R = \infty$ or $R > \sup [\|\mathbf{v}(t, \mathbf{y}^*)\| : t \geqslant 0, \|\mathbf{y}^*\| < H]$;

6) because $\mathbf{v}^* \leqslant \mathbf{v}^{**}$ we have $\mathbf{f}(t, \mathbf{v}^*) \leqslant \mathbf{f}(t, \mathbf{v}^{**})$ in region (3.9) and, furthermore, $\mathbf{f}(t, 0) \equiv 0$;

7) for $\|\mathbf{y}^*\| \to 0$ $\mathbf{v}(t, \mathbf{y}^*) \to 0$ uniformly with respect to $t \geqslant 0$. Let us consider the system of matching

$$\omega^\cdot = \mathbf{f}(t, \omega) \tag{3.10}$$

using the notation $\alpha = (\omega_1, \ldots, \omega_l)$.

Theorem 4. Let us assume that the zero solution of system (1.4) is exponentially asymptotically stable and that there exists a vector function $\mathbf{v}(t, \mathbf{y}^*)$ which satisfies conditions (1) — (7). If the zero solution of system (3.10) is uniformly α-stable with respect to t_0 when the conditions $\omega_{10} \geqslant 0, \ldots, \omega_{l0} \geqslant 0$ are satisfied, then the unperturbed motion of system (1.1) is uniformly Liapunov-stable and exponentially asymptotically x-stable for any functions $\mathbf{Y}$ and $\mathbf{X}$ that satisfy conditions (1.2) and (1.3).

Proof. The initial part of the proof of Theorem 1 up to and including the estimate (2.8) remains in this case unchanged. By virtue of (3.8), (3.6), (2.5), (2.7), (2.8) and (2.10) the derivative $d\mathbf{v}(t, \mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0))/dt$ satisfies the inequality

$$\begin{gathered} d\mathbf{v}(t, \mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0)) / dt < \mathbf{f}(t, \mathbf{v}(t, \mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0))) + \\ \varphi(t) e^{-\gamma(t-t_0)} (Mc_2 + c_3) \delta \mathbf{b}, \quad t \in (t_0, T) \end{gathered} \tag{3.11}$$

where $\mathbf{b} = (1, \ldots, 1)$. Let us consider the system of matching

$$\omega^{*\cdot} = \mathbf{f}(t, \omega^*) + \varphi(t) e^{-\gamma(t-t_0)} H\delta\mathbf{b}, \quad H = Mc_2 + c_3 \tag{3.12}$$

which corresponds to inequality (3.11).

We denote the solutions of systems (3.10) and (3.12) by $\omega(t;\ t_0,\ \omega_0)$ and $\omega^*(t;\ t_0,\ \omega_0^*)$, respectively, and show that the inequality

$$\omega^*(t;\ t_0,\ \omega_0) < \omega(t;\ t_0,\ \omega_0 + HD\delta\mathbf{b}) \tag{3.13}$$

is satisfied in every interval $(t_0,\ \tau)$, in which the solutions that appear in (3.13) are determinate. Let us assume the contrary, i.e. that the relationship

$$\omega^*(t;\ t_0,\ \omega_0) < \omega(t;\ t_0,\ \omega_0 + HD\delta\mathbf{b}), \quad t \in (t_0,\ t_1) \subset (t_0,\ \tau) \tag{3.14}$$

is valid and that for some i

$$\omega_i^*(t_1;\ t_0,\ \omega_0) = \omega_i(t_1;\ t_0,\ \omega_0 + HD\delta\mathbf{b}) \tag{3.15}$$

From (3.12), (3.14), (3.7) and the property (6) we obtain

$$\omega^*(t_1; t_0, \omega_0) = \omega_0 + \int_{t_0}^{t_1} \mathbf{f}(t, \omega^*(t;\ t_0, \omega_0))\, dt + H\delta\mathbf{b} \int_{t_0}^{t_1} \varphi(t)\, e^{-\gamma(t-t_0)}\, dt <$$

$$\omega_0 + HD\delta\mathbf{b} + \int_{t_0}^{t_1} \mathbf{f}(t, \omega(t;\ t_0, \omega_0 + HD\delta\mathbf{b}))dt = \omega(t_1; t_0, \omega_0 + HD\delta\mathbf{b})$$

which contradicts equality (3.15). Inequality (3.13) is thus proved.

Let an arbitrary ε, $0 < \varepsilon < \beta$ be specified (see (2.3)). By stipulation there exists $\lambda(\varepsilon) > 0$ such that for $\omega_{10} \geqslant 0, \ldots, \omega_{l0} \geqslant 0$ from

$$\sum_{s=1}^{m} |\omega_{s0}| < \lambda \tag{3.16}$$

follows

$$\sum_{s=1}^{l} |\omega_s(t;\ t_0, \omega_0)| < a(\varepsilon) \quad \text{for } t \geqslant t_0,\ t_0 \geqslant 0 \tag{3.17}$$

Using the number $\lambda(\varepsilon)$ it is possible by virtue of property (7) to select $\delta(\lambda(\varepsilon)) = \delta(\varepsilon) > 0$ such that the inequality

$$\|\mathbf{v}(t_0,\ \mathbf{y}_0)\| + HD\delta\, \|\mathbf{b}\| < \lambda \tag{3.18}$$

is satisfied in region (2.5).

Considering that condition (6) is stronger than that of Wazewski [7] and taking into account inequality (3.13), it can be readily shown, as in [8], that for $t \in (t_0,\ T)$ from (2.5) we obtain $\|\mathbf{y}(t;\ t_0,\ \mathbf{x}_0,\ \mathbf{y}_0)\| < \varepsilon$. Assuming that $\delta < c_2^{-1}\varepsilon$, we find from (2.7) that throughout the time interval during which conditions (2.6) are satisfied, the inequalities (2.14) are valid. Since $\varepsilon < \beta$, the inequalities (2.14) are valid for all $t \geqslant t_0$, Q. E. D.

Notes. (1) In the particular case of $m = 1$, $f_1 \equiv 0$ and $\varphi(t) = N = \text{const}$ from Theorem 4 we obtain the theorem of Dykhman [9]. We note in connection with this that in the general case of uniform stability it is not possible to prove the existence of a positive definite function v with a constantly negative derivative $v^\cdot$, which has bounded derivatives $\partial v / \partial y_i$.

2) It is possible to waive in Theorem 4 the smoothness of function v by substituting the weaker condition

$$\| \mathbf{v}(t, \mathbf{y}^*) - \mathbf{v}(t, \mathbf{y}^{**}) \| \leqslant \varphi(t) \| \mathbf{y}^* - \mathbf{y}^{**} \|$$

for condition (3.6). In that case $\mathbf{v}^{\cdot}$ is to be taken as the generalized derivative (see, e.g., [5, 10, 11]).

4. Let us consider the problem of instability.

Theorem 5. Let us assume that the zero solution of system (1.6) is unstable. Then the unperturbed motion of system (1.1) is y-unstable for any functions $\mathbf{Y}$ and $\mathbf{X}$ that satisfy condition (1.2).

Proof of this theorem is similar to that of Theorem 2 in [1].

The author thanks V. V. Rumiantsev for his interest in this work.

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Translated by J. J. D.